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Topological Solitons in the CP^N Model

Peng-Ming Zhang · Xi-Guo Lee · Shao-Feng Wu

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Abstract We investigate the solitons in the CP^N model in terms of the decomposition of gauge potential. Based on the ϕ-mapping topological current theory, the charge and position of solitons is determined by the properties of the typical component. Furthermore, the motion and the bifurcation of multi-soliton is discussed. And the knotted solitons in high dimension is explored also.

1 Introduction

The two-dimensional nonlinear field model have attracted a great deal of interest due to their important similarities to four-dimensional quantum chromodynamics. One of them is the CP^N model [1–4] where target space is given by the complex projective space $CP(N) = SU(N)/SU(N-1)\times U(1)$. The CP^N model is interesting in not only field theory but also the condensed matter physics, especially in the quantum Hall physics due to the exact solvability [5, 6]. Furthermore, in differential geometry, the soliton-solutions of CP^N models are known as harmonic maps [7], a rich industry of research on its own.

Among the important features of the model is the gauge structure of the composite Abelian gauge field $A_\mu = -iz^\dagger\partial_\mu z$ with a normalized CP^N spinor $z(x)$. With this dummy

P.-M. Zhang (✉) · X.-G. Lee
Institute of Modern Physics, Chinese Academy of Sciences, P.O. Box 31, Lanzhou 730000, China
e-mail: zhpm@impcas.ac.cn

P.-M. Zhang · X.-G. Lee
Center of Theoretical Nuclear Physics, National Laboratory of Heavy Ion Accelerator,
Lanzhou 730000, China

P.-M. Zhang
Department of Physics, Lanzhou University, Lanzhou 730000, China

S.-F. Wu
College of Science, Shanghai University, Shanghai 200444, China

gauge field the CP^N model

$$S=\frac{N}{2g^2}\int d\mathbf{x}|D_\mu z|^2, \qquad D_\mu z=(\partial_\mu - iA_\mu)z \tag{1}$$

is invariant under gauge transformation $z\to e^{i\alpha(x)}z$. Holding this gauge structure, many skills in gauge field can be employed to discuss the properties of the CP^N model. Recently, the decomposition theory of gauge potential is popular in studying confinement in gauge field theory [8–17] and the topological soliton [18–20]. By virtue of this theory, many interesting topological information is revealed even without the concrete solution. In this paper we will analyze the gauge field structure and the topological properties of the CP^N model, and topological charges and the motion of the solitons are discussed.

In next section, we review the CP^N model in brief. In Sect. 3, we give the decomposition of $U(1)$ gauge potential and discuss the soliton based on this decomposition. The moving and bifurcation of the solitons is studied based on the properties of the typical complex scalar field. Furthermore, the knotted solitons are considered in the high dimension in Sect. 4. At last, we present our conclusion.

2 CP^N Model and $U(1)$ Gauge Potential

Let us first recall the simplest case ($N=1$), i.e., the CP^1 model. It is well-known that the CP^1 model is equivalent to the $O(3)$ nonlinear sigma model which is governed by the action

$$I=\int d^3x\frac{1}{2\lambda^2}\partial_\mu n^a(x)\partial^\mu n^a(x), \quad \mu=0,1,2 \tag{2}$$

where λ is a coupling constant and the field $n^a(x)$ satisfies the constraint $|\vec{n}|^2=\sum_{n=1}^3(n^a)^2=1$. By employing the Hopf map $\vec{n}=\Psi^\dagger\vec{\sigma}\Psi$ and introducing the $U(1)$ gauge potential $a_\mu=-i\Psi^\dagger\partial_\mu\Psi$, the field action can be expressed as

$$I=\int d^3x\frac{2}{\lambda^2}(D_\mu\Psi)^\dagger(D^\mu\Psi), \tag{3}$$

where the covariant derivative $D_\mu=\partial_\mu - ia_\mu$.

Generalized to the higher order, we have the CP^N model (1) defined by the $(N+1)$-component complex field $z^T=(z_0,z_1,\dots,z_N)$ [7]. Here the action is still invariant under local $U(1)$ transformations

$$z_a(x)\to z_a(x)e^{i\alpha(x)}, \quad a=0,1,2,\dots,N \tag{4}$$

where $\alpha(x)$ can be an arbitrary real function of x, but the same for all the components a. From the viewpoint of geometry, any $U(1)$ gauge potential can be regarded as the connection on principal bundle $P(M,U(1),\pi)$ on the d-dimensional base manifold M, with its covariant derivative given by $D_\mu\psi=\partial_\mu\psi - iA_\mu\psi$ ($\mu=0,1,2$), where ψ is a section of the principal bundle. In the $U(1)$ gauge structure, it is a complex function $\psi=\phi^1+i\phi^2$ where (ϕ^1,ϕ^2) forms a 2-dimensional real vector. In this model, the complex field ψ can be chosen as one component of spinor $z(x)$.

However, one should notice that only N components are independent since the spinor $z(x)$ is normalized $z(x)^\dagger z(x)=1$. As we know, the topological property can be understood in the distance due to the Stokes' theorem. For the CP^N soliton, in the distance the nontrivial components should possess the similar topological property since they describe the same object. Therefore, it is enough to choose arbitrarily one of those nontrivial components as the phototype complex scalar field to discuss the soliton. Without loss the generality we may choose $w(x)$ $(l = 1, 2, \dots, N)$ as the phototype nontrivial component which satisfies

$$D_\mu w = \partial_\mu w - i A_\mu w, \quad w = w^1 + i w^2. \tag{5}$$

With the decomposition theory of gauge potential, the $U(1)$ gauge potential can be expressed with the real field [21] $A_\mu = \varepsilon_{AB}\partial_\mu m^A m^B + \partial_\mu\theta(A, B = 1, 2)$, where θ is only a phase factor and $m^A = w^A/\sqrt{w^B w^B}$ are the unit vector. Here one should notice that this result is limited to discuss the topological properties since it is exact only in the distance. For an arbitrary normalized constant spinor z_0 we can always rotate it to be one of the following forms $(1, 0, \dots, 0)^T, (0, 1, 0, \dots, 0)^T, \dots, (0, 0, \dots, 0, 1)^T$, which share the same topological information. So we may choose one of them to discuss the topology in the CP^N model. The different choice just means the different direction of spinor in the vacuum. In the condensed matter physics (for example spinor BEC), such choice is dependent of the configuration of ground state.

The soliton current is defined as

$$J^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda, \quad \mu = 0, 1, 2. \tag{6}$$

With the decomposition of gauge potential one can obtain

$$J^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\varepsilon_{AB}\partial_\nu m^A \partial_\lambda m^B, \tag{7}$$

which is the standard form of the $(2 + 1)$-dimensional topological current used in many fields [18, 19, 22, 23]. One should notice that the above current is conserved for the time-dependent solution comparing to the traditional soliton current which is conserved only for the static case. Therefore the current (7) is more powerful to study the moving CP^N solitons.

Before we discuss the moving CP^N solitons, the topological charge should be checked and compared with the traditional definition [7]. With the current (7) the topological charge of CP^N solitons is

$$Q = \int J^0 d\mathbf{x} = \frac{1}{2\pi}\int d^2x \varepsilon_{\mu\nu}\varepsilon_{AB}\partial_\mu m^A \partial_\nu m^B \tag{8}$$

which should be equivalent to the following definition [7]

$$Q = \frac{1}{2\pi i}\int d^2x \varepsilon_{\mu\nu}\partial_\mu z^\dagger \partial_\nu z. \tag{9}$$

It is well-known at the distance the spinor z of one soliton has the form

$$z = z^{(0)} e^{in\varphi} \tag{10}$$

where $z^{(0)}$ is any fixed complex vector with $(z^{(0)})^\dagger z^{(0)} = 1$. Thus the topological charge is just the winding number

$$Q = n.$$

On the other hand, the nontrivial component w also share the form (10) at the distance

$$w = |w|e^{in\varphi}, \tag{11}$$

then the topological charge will be recovered from (8)

$$Q = \frac{1}{2\pi}\int_{r\to\infty} d\varphi(m^1\partial_\varphi m^2 - m^2\partial_\varphi m^1) = n. \tag{12}$$

Therefore the topological information of single soliton is well revealed with the topological current (7). However the traditional definition is limited in the static case and single soliton and the rich topological structure of the moving multi-soliton is hidden. In the following we will discuss the topological structure of the moving CP^N solitons.

3 Topological Structure of CP^N Solitons

By making use of the ϕ-mapping topological current theory [18, 19, 22, 23], the current (7) have the δ-function form which make the defect clear

$$J^\mu = \delta^2(\vec{w})D^\mu\left(\frac{w}{x}\right), \tag{13}$$

where $D^\mu(w/x) = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\varepsilon_{AB}\partial_\nu w^A\partial_\lambda w^B$ is the Jacobian vector. It is obvious that the solitons always appear in the zeroes of the $\vec{w}$, so it is necessary to study the zero points of $\vec{w}$ to determine the non-zero solutions of J^μ. One can find that the general solutions of $\vec{w}(x^1, x^2, t) = 0$ can be expressed as $x^1 = r_l^1(t)$, $x^2 = r_l^2(t)$ under the regular condition $D^0(w/x) \neq 0$, which represent the worldlines L_l of lth moving isolated soliton $\vec{r}_l(t)$ in $(2+1)$-dimensional space-time. The location of lth soliton is determined by the lth zero point $\vec{r}_l(t)$.

In δ-function theory [24] and ϕ-mapping topological current theory, one can prove that

$$\delta^2(\vec{w}) = \sum_{l=1}^{N}\frac{\beta_l}{|D(\frac{w}{x})|_{\vec{r}_l}}\delta^2(\vec{x} - \vec{r}_l), \tag{14}$$

for a fixed time t. The positive integer β_l is the Hopf index of w-mapping [25], which means that when $\vec{r}$ covers the neighborhood of the zero point $\vec{r}_l(t)$ once, the vector field $\vec{w}$ covers the corresponding region in w-space β_l times. Meanwhile the direction vector of soliton is given by [19]

$$v^i = \left.\frac{dx^i}{dt}\right|_{\vec{r}_l} = \left.\frac{D^i(w/x)}{D(w/x)}\right|_{\vec{r}_l}, \qquad v^0 = 1. \tag{15}$$

Then from (14) and (15) we obtain the inner structure of J^μ:

$$\vec{J} = \sum_{l=1}^{N} W_l\vec{v}_l\delta^2(\vec{x} - \vec{r}_l(t)), \qquad \rho = j^0 = \sum_{l=1}^{N} W_l\delta^2(\vec{x} - \vec{r}_l(t)), \tag{16}$$

where $W_l = \beta_l\eta_l$ is the winding number of $\vec{w}$ around the solitons with $\eta_k = sgn D(w/x)_{\vec{r}_j} = \pm 1$ being the Brouwer degree of w-mapping [26]. The above current shows the movement of the solitons in space-time from the viewpoint of topology without any concrete model. It is easy to check the topological charge of multi-solitons

$$Q = \int \rho d^2x = \sum_{l=1}^{N} W_l = \sum_{l=1}^{N} \beta_l \eta_l .$$

Furthermore the dynamics of moving soliton can be explored qualitatively in terms of the property of complex scalar field w.

From (16) one can find that the zeros of scalar field w play an important role in studying the CP^N solitons, i.e. the solutions of the following equation

$$w^1(x^1, x^2, t) = 0, \qquad w^2(x^1, x^2, t) = 0. \tag{17}$$

As we know before, if the Jacobian

$$D\left(\frac{w}{x}\right) = \frac{\partial(w^1, w^2)}{\partial(x^1, x^2)} \neq 0, \tag{18}$$

we will have the isolated solutions $x^1 = x^1(t), x^2 = x^2(t)$ of (17). However, when the condition (18) fails, the usual implicit function theorem is of no use. The above results will charge in some way and will lead to the branch process. We denote one of the zero points as $(t^*, \vec{z}_l)$. If the Jacobian $D^1(\frac{w}{x})|_{(t^*,\vec{z}_l)} \neq 0$, we can use the Jacobian $D^1(\frac{w}{x})$ instead of $D(\frac{w}{x})$ for the purpose of using the implicit function theorem. Then we have an unique solution of (17) in the neighborhood of the points $(t^*, \vec{z}_l)$

$$t = t(x^1), \qquad x^2 = x^2(x^1), \tag{19}$$

with $t^* = t(z_l^1)$. And we call the critical points $(t^*, \vec{z}_l)$ the limit points. In the present case, it is easy to know that

$$\left.\frac{dx^1}{dt}\right|_{(t^*,\vec{z}_l)} = \frac{D^1(w/x)|_{(t^*,\vec{z}_l)}}{D(w/x)|_{(t^*,\vec{z}_l)}} = \infty, \quad \text{or} \quad \left.\frac{dt}{dx^1}\right|_{(t^*,\vec{z}_l)} = 0. \tag{20}$$

The Taylor expansion of the solution of (19) at the limit point $(t^*, \vec{z}_l)$ is [20]

$$t - t^* = \frac{1}{2} \left.\frac{d^2t}{(dx^1)^2}\right|_{(t^*,\vec{z}_l)} (x^1 - z_l^1)^2 \tag{21}$$

which is a parabola in the $x^1 - t$ plane. From (21), we can obtain two solutions $x_1^1(t)$ and $x_2^1(t)$, which give two branch solutions (worldlines of vortices) of (17). If $\frac{d^2t}{(dx^1)^2}|_{(t^*,\vec{z}_l)} > 0$, we have the branch solutions for $t > t^*$, otherwise, we have the branch solutions for $t < t^*$. These two cases are related to the origin and annihilation of vortices, and one will find that the velocity of solitons is infinite when they are annihilating or generating, which is gained only form the topology of the scalar field. Since the topological charge of solitons is identically conserved (7), the topological index of two solitons must be opposite at the limit point, i.e., $\eta_{l_1} = -\eta_{l_2}$.

Furthermore, if we require $D^1(\phi/x)|_{(t^*,\vec{z}_l)} = 0$ beside $D(\phi/x)|_{(t^*,\vec{z}_l)} = 0$, which means that the function relationship between t and x^1 is not unique in the neighborhood of the bifurcation point $(t^*, \vec{z}_l)$, i.e., the velocity field of solitons is indefinite at the point $(t^*, \vec{z}_l)$.

Next, we will find a simple way to search for the directions of all branch curves (or velocity field of solitons) at the bifurcation point. Assume that the bifurcation point $(t^*, \vec{z}_l)$ has been found from, then the Taylor expansion of the solution in the neighborhood of the bifurcation point $(t^*, \vec{z}_l)$ can be expressed as [20]

$$A(x^1 - z_1^1)^2 + 2B(x^1 - z_1^1)(t - t^*) + C(t - t^*)^2 = 0, \tag{22}$$

which leads to

$$A\left(\frac{dx^1}{dt}\right)^2 + 2B\frac{dx^1}{dt} + C = 0, \tag{23}$$

and

$$C\left(\frac{dt}{dx^1}\right)^2 + 2B\frac{dt}{dx^1} + A = 0, \tag{24}$$

where A, B and C are three parameters determined by the configuration of typical component around the solitons. The solutions of (23) or (24) give different directions of the branch curves (worldlines of solitons) at the bifurcation point.

The remainder component dx^2/dt can be given by

$$\frac{dx^2}{dt} = f_1^2\frac{dx^1}{dt} + f_t^2$$

where partial derivative coefficients f_1^2 and f_t^2 have been calculated [20]. From these relations we find that the values of dx^2/dt at the bifurcation point $(t^*, \vec{z}_l)$ are also possible different because (24) may give different values of dx^1/dt. The above solutions reveal the evolution of solitons. Besides the encountering of the solitons, i.e., two solitons encounter and then depart at the bifurcation point along different branch point, it may split into several solitons along different branch curves. On the contrary, several solitons can merge into one soliton at the bifurcation point. The identical conversation of the topological charge shows the sum of the topological charge of final solitons must be equal to that of the initial solitons at the bifurcation point, i.e.,

$$\sum_f \beta_{j_f}\eta_{j_f} = \sum_i \beta_{j_i}\eta_{j_i}, \tag{25}$$

for fixed j. Furthermore, from above studies we see that the generation, annihilation and bifurcation of solitons are not gradual charges, but start at a critical value of arguments, i.e. a sudden charge.

4 Knotted Solitons in the CP^N

In this section, we discuss the closed linear topological object, knot, in the CP^N model. It is easy to prove that in the 3-dimensional space the soliton current can be expressed as

$$J^\mu = \sum_{l=1}^{N} W_l \int_{L_1} \frac{dx^\mu}{ds}\delta^3(\vec{x} - \vec{r}_l)ds. \tag{26}$$

Using (26), Hopf number becomes

$$H = \int A_\mu J^\mu d^3x = \sum_{l=1}^{N} W_l \int_{L_l} A_\mu dx^\mu. \tag{27}$$

It can be seen that when these N vortex lines are N closed curves, i.e., a family of N knots γ_l $(l = 1, \ldots, N)$, (27) leads to

$$H = \sum_{l=1}^{N} W_l \oint_{\gamma_l} A_\mu dx^\mu = c_1^2, \tag{28}$$

which indicates that the Hopf number describes the knots and is a topological invariant [27]. Consider the $U(1)$ gauge transformation of A_μ [28]:

$$A'_\mu = A_\mu + \partial_\mu \alpha, \tag{29}$$

where $\alpha \in \mathbf{R}$ is a phase factor denoting the $U(1)$ transformation. It is seen that the $(\partial_\mu \alpha)$ term in (29) contributes nothing to the integral H, hence the expression (28) is invariant under the gauge transformation. Therefore one can conclude that H is a topological invariant for the knotlike vortex lines [29]. Furthermore, it is proven that [30]

$$H = 2\pi \left[\sum_{k=1}^{N} W_k^2 SL(\gamma_k) + \sum_{k,l=1\,(k\neq l)}^{N} W_k W_l Lk(\gamma_k, \gamma_l) \right], \tag{30}$$

where $SL(\gamma_k)$ is the self-linking number of knot γ_k and $Lk(\gamma_k, \gamma_l)$ is the linking number of two knots γ_k and γ_l. This result shows that the Hopf number describes the linking number of knots correctly [29, 31–36]. Since the self-linking and the linking numbers are both the invariant characteristic numbers of the knotlike closed curves in topology, I is an important invariant required to describe the knotlike vortex lines in CP^N model.

5 Conclusion

In this paper, we mainly discussed the different topological defects in the CP^N model. Firstly, the point-like CP^N soliton, characterized by the first Chern class, is discussed and the corresponding inner topological structure is presented. By virtue of the ϕ-mapping topological current theory, we find that the solitons are created from the zero points of the typical component of the spinor field z. And the bifurcation of solitons is discussed without the concrete solutions. Secondly another topological object, knot which is famous in the field of physics and mathematics, is considered in the high dimensional spacetime. We show that this Hopf invariance is just the linking number of many knots in the system in the 3-dimensional space. At last, we want to point out that these topological structures can all be characterized by the ϕ-mapping topological numbers–Hopf indices and Brouwer degrees, and their locations and motions can be rigorously determined by ϕ-mapping topological current theory.

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